NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 2

2015 HSC Assessment Task 1 December 2014

Name: ____________________________ **Mathematics Class:** 12MZ___

Student Number: ________________

Time Allowed: **55 minutes + 2 minutes reading time**

Available Marks: **38*i***

Instructions:

- Questions are *not* of equal value.
- Start each question in a new booklet.
- Show all necessary working.
- Do not work in columns.
- Marks may be deducted for incomplete or poorly arranged work.

Question	1-5	6	7ab	7cd	Total
E2				/5*i*	/5*i*
E3	/5*i*	/18*i*	/10*i*		/33*i*
					/38*i*

Section I – Multiple Choice

5i Marks
Attempt question 1−5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1−5

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1. Which pair of statements is correct?

(A) $|-2-3i| = 13$ and $-\pi < \arg(-2-3i) < -\dfrac{\pi}{2}$

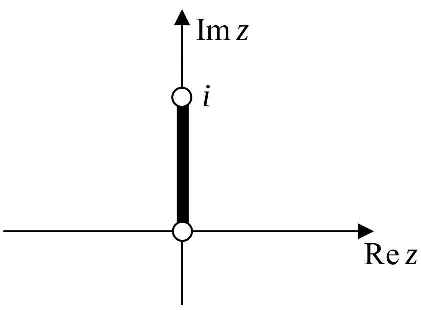
(B) $|-2-3i| = 13$ and $-\dfrac{\pi}{2} < \arg(-2-3i) < 0$

(C) $|-2-3i| = \sqrt{13}$ and $-\pi < \arg(-2-3i) < -\dfrac{\pi}{2}$

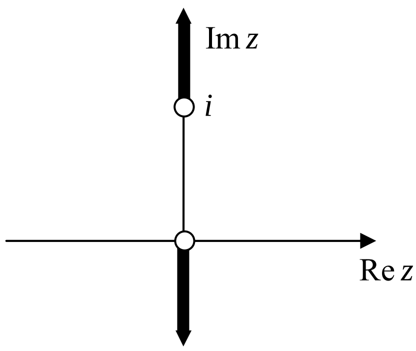
(D) $|-2-3i| = \sqrt{13}$ and $-\dfrac{\pi}{2} < \arg(-2-3i) < 0$

2. Which of the following loci is represented by the equation $\arg z = \arg i$?

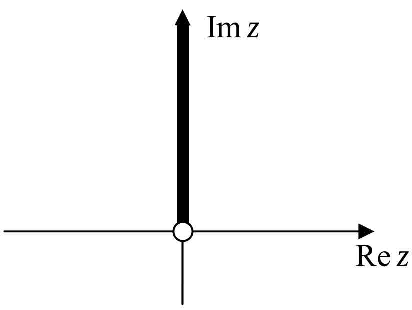
(A)



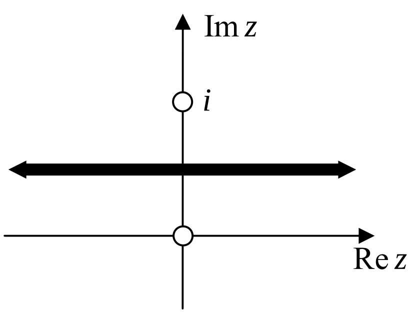
(B)



(C)



(D)



3. If z is a complex number where $z \neq 0$, which one of the following statements is *not necessarily* true?

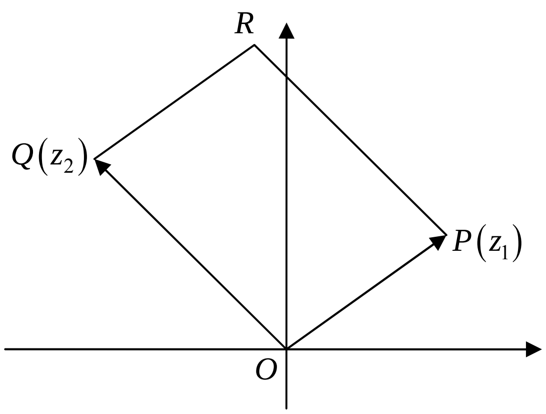
(A) $|z|^2 = (\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2$

(B) $\arg z = \tan^{-1}\left(\dfrac{\mathrm{Im}\, z}{\mathrm{Re}\, z}\right)$

(C) $\mathrm{Re}\left(z^2\right) \leq (\mathrm{Re}\, z)^2$

(D) $\arg z + \arg \overline{z} = 0$

4. In the following diagram, $OPRQ$ is a parallelogram, and the vectors OP and OQ represent the complex numbers z_1 and z_2 respectively.



NOT TO SCALE

If $\dfrac{z_2}{z_1} = \sqrt{3}\, i$, what is the value of $\left|\dfrac{z_1 + z_2}{z_2 - z_1}\right|$?

(A) 1

(B) $\sqrt{3}$

(C) $\dfrac{1}{\sqrt{3}}$

(D) $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

5. If $z = \omega$ is a non-real solution of the equation $z^3 = 1$, which of the following expressions is equal to $\left(1 + \omega - \omega^2\right)^7$?

(A) 128ω

(B) -128ω

(C) $128\omega^2$

(D) $-128\omega^2$

Section II

33*i* Marks
Attempt Questions 6 and 7
Allow about 47 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing sheets are available.

In Questions 6 and 7, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (18*i* marks) Use a SEPARATE writing booklet.

(a) Let $z = 2 + 3i$ and $\omega = 1 + i$.

Find in the form $a + ib$:

(i) $z\omega$ **2**

(ii) $\dfrac{1}{\omega}$ **2**

(b) Solve the equation $x^2 - ix + 2 = 0$, writing your solutions in the form $x = ki$, where k is real. **2**

(c) Solve for z: **2**

$$z + 3i\,\overline{z} = -1 + 13i$$

(d) Sketch the locus of the complex number z defined by $\arg z = \pi$ AND $|z+1| < 2$. **2**

[Make clear what your final answer is, using words if necessary.]

(e) Let $z = -1 + i\sqrt{3}$.

(i) Write z in modulus-argument form. **1**

(ii) Hence evaluate $z^5 + 16z$ in the form $a + ib$. **2**

(iii) If $\omega = \dfrac{z}{c}$ where c is a positive real number, for what value of c is it possible for the equation $\omega^n = \omega$ to have solutions other than $n = 1$? **1**

(iv) For this value of c, write down a general solution for n to the equation $\omega^n = \omega$. **1**

Question 6 (continued)

(f) If $z = r(\cos\theta + i\sin\theta)$ where $r > 0$, find r and the smallest positive value of θ such that $2z^2 = 9 + 3\sqrt{3}\,i$. **3**

Question 7 on next page

Question 7 (15 marks) Use a SEPARATE writing booklet

(a) (i) Find a square root of $5-12i$ in the form $a+ib$. **2**

(ii) Hence sketch the locus on the Argand diagram represented by the equation **3**

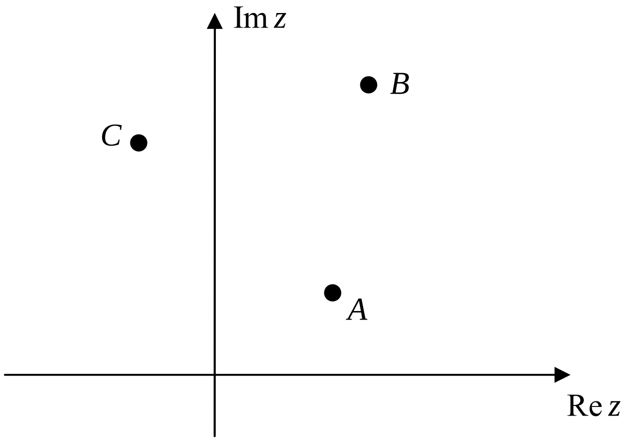
$$\left|z^2-5+12i\right| = \left|z-3+2i\right|$$

(b) (i) Show that the locus of complex numbers z satisfying **2**

$\left|z-(1+i)\right| = \mathrm{Re}\left[z-(-1+i)\right]$ is the parabola $(y-1)^2=4x$.

(ii) Find the range of values of $\arg z$ for points on this locus. **3**

(c) The points A, B and C, representing non-zero complex numbers a, b and c, are the vertices in anti-clockwise order of an equilateral triangle in the Argand diagram. **2**



By considering vectors AB and AC, show that the complex numbers a, b and c satisfy

$$2c = (a+b) + i\sqrt{3}(b-a)$$

(d) Certain complex numbers z satisfy the relation $cz = i(z-\omega)$ where $\omega = 1+i$ and $c>0$ is a real number which is free to vary. **3**

On an Argand diagram, sketch the locus of all such complex numbers z.

A sketch only, without justification or development of the locus, will not earn full credit.

End of Paper

2015 Extension 2 Assessment 1 Solutions

Section I – Multiple Choice

1. C **2.** C **3.** B **4.** A **5.** D

Working:

1. $|-2-3i|^2 = (-2)^2+(-3)^2 = 13$

$|-2-3i| = \sqrt{13}$

$-2-3i$ is in the 3rd quadrant, so $-\pi < \arg(-2-3i) < -\dfrac{\pi}{2}$ **[C]**

2. $\arg z = \arg i$ is the same as $\arg z = \dfrac{\pi}{2}$ **[C]**

Note: (A) is $\arg z = \arg(z-i)+\pi$ (B) is $\arg z = \arg(z-i)$ (D) is $|z| = |z-i|$

3. (A) $|z|^2 = x^2+y^2$ true

(C) $\text{Re}(z^2) = \text{Re}\left[(x^2-y^2)+2ixy\right]$

$= x^2-y^2$

$\le x^2$

$= (\text{Re}\, z)^2$ true

(D) If $\arg z = \theta$ then $\arg \overline{z} = -\theta$ true

(B) $\tan(\arg z) = \dfrac{y}{x}$ does NOT imply $\arg z = \tan^{-1}\left(\dfrac{y}{x}\right)$ as there are TWO arguments whose tan is $\dfrac{y}{x}$

[B]

4. If $\dfrac{z_2}{z_1} = \sqrt{3}\, i$ then $\angle POQ = 90°$, so $OPRQ$ is actually a rectangle.

$$\left|\frac{z_1+z_2}{z_2-z_1}\right| = \frac{|z_1+z_2|}{|z_1-z_2|} = \frac{OR}{PQ} = 1 \quad \text{(since the diagonals of a rectangle are equal)}$$

(The $\sqrt{3}$ was a red herring, and doesn't change the answer) **[A]**

Alternatively $\left|\dfrac{z_1+z_2}{z_2-z_1}\right| = \left|\dfrac{1+\frac{z_2}{z_1}}{\frac{z_2}{z_1}-1}\right| = \left|\dfrac{1+i\sqrt{3}}{i\sqrt{3}-1}\right| = \dfrac{|1+i\sqrt{3}|}{|i\sqrt{3}-1|} = \dfrac{2}{2} = 1$

5. Since $1+\omega+\omega^2 = 0$, then $1+\omega-\omega^2 = -2\omega^2$.

$$(1+\omega-\omega^2)^7 = (-2\omega^2)^7 = -128\omega^{14} = -128(\omega^3)^4\cdot\omega^2 = -128\omega^2 \quad (\text{since } \omega^3 = 1)$$

[D]

Section II – Free Response

Question 6

(a)

Let $z = 2+3i$ and $\omega = 1+i$.
Find in the form $a+ib$:

(i)

$z\omega$ **2**

$$z\omega = (2+3i)(1+i)$$
$$= 2+2i+3i-3$$
$$= -1+5i$$

(ii)

$\frac{1}{\omega}$ **2**

$$\frac{1}{\omega} = \frac{1}{1+i} \times \frac{1-i}{1-i}$$
$$= \frac{1-i}{1+1}$$
$$= \frac{1}{2} - \frac{1}{2}i$$

(b)

Solve the equation $x^2 - ix + 2 = 0$, writing your solutions in the form $x = ki$, where k is real. **2**

Factorise
$(x-2i)(x+i)=0$
$x = 2i, -i$

OR

Let the roots be α and β.
$\alpha + \beta = i$ and $\alpha\beta = 2$
By inspection $\alpha = 2i$ and $\beta = -i$

OR

$$x = \frac{i \pm \sqrt{(-i)^2 - 4(1)(2)}}{2}$$
$$= \frac{i \pm \sqrt{-9}}{2}$$
$$= \frac{i \pm 3i}{2}$$
$$x = -i,\ 2i$$

OR

$$x^2 - ix + \left(-\frac{i}{2}\right)^2 = -2 + \left(-\frac{i}{2}\right)^2$$
$$\left(x - \frac{i}{2}\right)^2 = -2 - \frac{1}{4} = -\frac{9}{4}$$
$$x - \frac{i}{2} = \pm \frac{3}{2} i$$
$$x = \frac{i}{2} \pm \frac{3i}{2}$$
$$x = -i,\ 2i$$

Marker Comments

Part (a) and (b) were generally well done.

Question 6 (continued)

(c)

Solve for z: $z + 3i\bar{z} = -1 + 13i$ **2i**

Let $z = x + iy$

$$x + iy + 3i(x - iy) = -1 + 13i$$
$$(x + 3y) + i(3x + y) = -1 + 13i$$

Equating real and imaginary parts

$$x + 3y = -1$$
$$3x + y = 13$$

Solving simultaneously: $x = 5,\ y = -2$

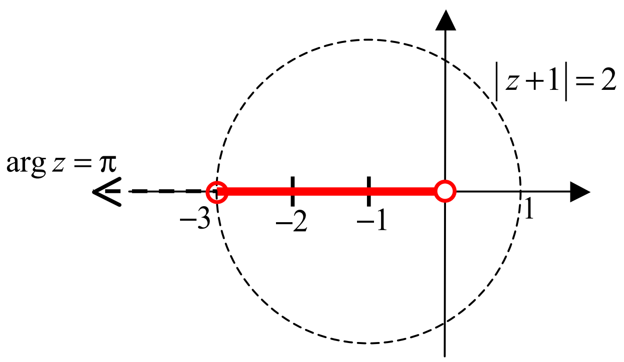
$\therefore\ z = 5 - 2i$

Marker Comments

Most students knew how to do this part. Once x and y are found, students are advised to finalise their answer and state what z is. There was no penalty applied for not doing this, but it is good practice.

(d)

Sketch the locus of the complex number z defined by $\arg z = \pi$ AND $|z+1| < 2$. **2i**
[Make clear what your final answer is, using words if necessary.]



The final locus is marked in **red**

Marker Comments

Most students knew the two individual loci but many were confused about how to combine the two.
Only one or two students thought to offer their final answer in a SEPARATE diagram to avoid any confusion.
Several students knew that the point (–3,0) was not included in the locus but failed to mark this on their sketch and incurred a small penalty.
The answer required was a sketch of the locus – the accompanying words were only for clarification.

Question 6 (continued)

(e)

Let $z = -1 + i\sqrt{3}$.

(i)

Write z in modulus-argument form. *i*

$$z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

(ii)

Hence evaluate $z^5 + 16z$ in the form $a + ib$. *2i*

$$\begin{aligned} z^5 + 16z &= 2^5\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^5 + 16\times 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \\ &= 32\left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right) + 32\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \\ &= 32\left(\cos\frac{2\pi}{3} - \cancel{i\sin\frac{2\pi}{3}} + \cos\frac{2\pi}{3} + \cancel{i\sin\frac{2\pi}{3}}\right) \\ &= 64\cos\frac{2\pi}{3} \\ &= -32 \end{aligned}$$

OR

$$\begin{aligned} z^5 + 16z &= \left(2\operatorname{cis}\tfrac{2\pi}{3}\right)^5 + 16\left(2\operatorname{cis}\tfrac{2\pi}{3}\right) \\ &= 32\operatorname{cis}\tfrac{10\pi}{3} + 32\operatorname{cis}\tfrac{2\pi}{3} && \text{Using DeMoivre's Theorem} \\ &= 32\left(\overline{\operatorname{cis}\tfrac{2\pi}{3}} + \operatorname{cis}\tfrac{2\pi}{3}\right) && \left[\because \operatorname{cis}\tfrac{10\pi}{3} = \operatorname{cis}\left(-\tfrac{2\pi}{3}\right)\right] \\ &= 32\left(2\cos\tfrac{2\pi}{3}\right) && \left[\because w + \overline{w} = 2\operatorname{Re}(w)\right] \\ &= -32 \end{aligned}$$

Marker Comments

A small number of students wrote down an incorrect argument in part (i).
Use the POL key on the calculator if in doubt.
Part (ii) was mostly well done. Some students factorized $z^5 + 16z = z\left(z^4 + 16\right)$ which offered no benefit and if anything made the calculations more tedious.
Students who converted the argument of z^5 to a principal argument could simplify more easily by using sum of conjugates. Some students did not leave their answer in $a + ib$ form which is what was required.

(iii)

If $\omega = \dfrac{z}{c}$ where c is a positive real number, for what value of c is it possible for the equation $\omega^n = \omega$ to have solutions other than $n = 1$? *i*

If $\omega = \omega^n$ then $|\omega| = 1$. Therefore, $\left|\dfrac{z}{c}\right| = 1$ and $c = |z| = 2$. Note $c > 0$ so $c \neq -2$.

Question 6 (continued)

(e) (iv)

For this value of c, write down a general solution for n to the equation $\omega^n = \omega$. *i*

$\arg(\omega) = \arg(\omega^n)$ Therefore, $\frac{2n\pi}{3} = \frac{2\pi}{3} + 2k\pi,\ k \in \mathbb{Z}$

$2n\pi = 2\pi + 6k\pi$

$n = 1 + 3k$

$\therefore n = 3k+1$ where k is an integer

Marker Comments

Part (iii) and (iv) were poorly done. Most made a start on part (iii) but did not comprehend that solutions are required for c not n.
Some students did not make the link that the z in this question was the same z as in part (i).
Recognising that $\omega = \omega^n$ requires $|\omega| = 1$ was the key although not many used this.
Those who successfully answered (iii) were usually able to answer (iv) as well.

(f)

If $z = r(\cos\theta + i\sin\theta)$ where $r > 0$, find r and the smallest positive value of θ such that $2z^2 = 9 + 3\sqrt{3}\,i$. *3i*

$$9 + 3\sqrt{3}i = \sqrt{108}\ \text{cis}\frac{\pi}{6}$$

$$2r^2\text{cis}2\theta = \sqrt{108}\ \text{cis}\frac{\pi}{6}$$

$2r^2 = \sqrt{108}$

$r^2 = \sqrt{27}$

$r = \sqrt[4]{27}$

and

$2\theta = \frac{\pi}{6} + 2k\pi,\ k \in \mathbb{Z}$

$\theta = \frac{\pi}{12} + k\pi$

For smallest positive argument, $\theta = \frac{\pi}{12}$

ALTERNATIVELY

$|2z^2|^2 = 9^2 + (3\sqrt{3})^2$

$= 108$

$|2z^2| = \sqrt{108} = 6\sqrt{3}$

$|z^2| = 3\sqrt{3} = 27^{\frac{1}{2}}$

$r = |z| = 27^{\frac{1}{4}}$

$\arg(z^2) = \arg(2z^2)$ $[\because \arg 2 = 0]$

$= \tan^{-1}\frac{3\sqrt{3}}{9}$ $[\text{first quadrant}]$

$= \tan^{-1}\frac{1}{\sqrt{3}}$

$= \frac{\pi}{6}$

$\theta = \arg z = \frac{\pi}{12}$

Marker Comments

Those who used mod-arg form had success.

Some students expanded $(\cos 2\theta + i\sin 2\theta)^2$ and then used double angle formulae which was laborious and

Question 7

(a) (i)

Find a square root of $5-12i$ in the form $a+ib$. **2**

Let $\sqrt{5-12i} = a+ib$ $\quad (a, b \text{ real})$

$$5-12i = (a+ib)^2$$
$$= (a^2-b^2) + 2abi$$

Method 1: Equating real and imaginary parts

$a^2-b^2=5$ ①

$ab=-6$ ②

Now $(a^2+b^2)^2 = (a^2-b^2)^2 + (2ab)^2$

$\therefore (a^2+b^2)^2 = 5^2+12^2$

$= 169$

$\therefore a^2+b^2 = 13$ ③

① + ③: $2a^2 = 18$

$\therefore a^2 = 9$

$\therefore a = \pm 3$

Using ②: If $a = 3$, $b = -2$

$\therefore$ the square roots of $5-12i$ are $\pm(3-2i)$

Method 2: Inspection

$5-12i = 5-2\times 3\times 2i$

$= 3^2-2^2-2(3\times 2i)$

$= (3-2i)^2$

$\therefore$ the square roots of $5-12i$ are $\pm(3-2i)$

Method 3:

$a^2-b^2=5$ ① $\quad ab=-6$ (equating real and imaginary parts)

$b = -\dfrac{6}{a}$ ②

Sub ② in ①: $a^2 - \left(-\dfrac{6}{a}\right)^2 = 5$

$a^4 - 5a^2 - 36 = 0$

$(a^2-9)(a^2+4) = 0$

$a = \pm 3, b = \mp 2$

$\therefore$ the square roots of $5-12i$ are $\pm(3-2i)$

Marker Comments

This part was generally well done

Question 7 (continued)

(a) (ii)

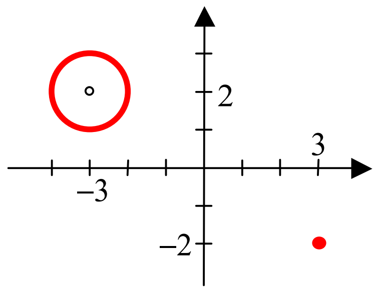
Hence sketch the locus on the Argand diagram represented by the equation ***3i***

$$\left|z^2-5+12i\right| = \left|z-3+2i\right|$$

$$\left|z^2-5+12i\right|=\left|z-3+2i\right|$$
$$\left|z^2-(5-12i)\right|=\left|z-(3-2i)\right|$$
$$\left|\left(z-\sqrt{5-12i}\right)\times\left[\left(z+\sqrt{5-12i}\right)\right]\right|=\left|z-(3-2i)\right|$$
$$\left|z-(3-2i)\right|\times\left|z+(3-2i)\right|=\left|z-(3-2i)\right|$$
$$\left|z-(3-2i)\right|\times\left|z-(-3+2i)\right|=\left|z-(3-2i)\right|$$

You can only divide by $\left|z-(3-2i)\right|$ if you *separately* account for the fact that it can be zero.

So solution is $\left|z-(-3+2i)\right| = 1$ OR $z=3-2i$



Marker Comments

This was a "Hence" question and too many students didn't consider this.

Too many students assumed that by seeing | … | = | … | that the locus HAD to be a perpendicular bisector.

Generally, if a student saw the connection to (a) (i), they were successful in finding the locus EXCEPT for the problem of cancelling.
Cancelling $\left|z-(3-2i)\right|$ without considering whether it could be zero was a 1-mark penalty.

Another problem is that too many students are not taking care with circle loci in getting the centre right i.e. too many students when faced with $\left|z+3-2i\right|$, did not re-write it as $\left|z-(-3+2i)\right|$.
This meant they drew a circle with centre (3, –2) instead of (–3, 2). This was penalised.

Question 7 (continued)

(b) (i)

Show that the locus of complex numbers z satisfying **2i**

$\left|z-(1+i)\right| = \text{Re}\left[z-(-1+i)\right]$ **is the parabola** $(y-1)^2 = 4x$.

Let $z = x+iy$

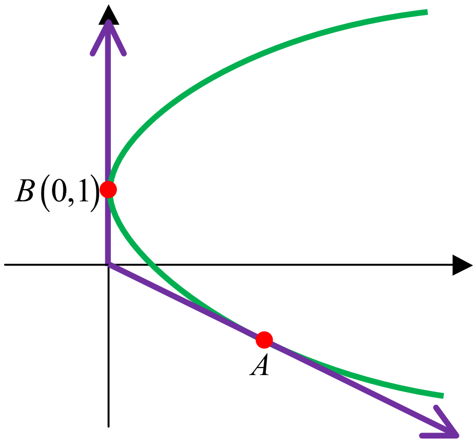
$$
\begin{aligned}
\left|x+iy-1-i\right| &= \text{Re}\left[x+iy+1-i\right] \\
\left|(x-1)+i(y-1)\right| &= \text{Re}\left[(x+1)+i(y-1)\right] \\
\left|(x-1)+i(y-1)\right| &= x+1 \\
\left|(x-1)+i(y-1)\right|^2 &= (x+1)^2 \\
(x-1)^2+(y-1)^2 &= (x+1)^2 \\
(y-1)^2 &= (x+1)^2-(x-1)^2 \\
&= \left[(x+1)+(x-1)\right]\left[(x+1)-(x-1)\right] \\
&= 2x \times 2 \\
(y-1)^2 &= 4x
\end{aligned}
$$

Marker Comments

This was generally well done.
Several students used the locus definition of a parabola to establish the required locus equation

Question 7 (continued)

(b) (ii) **Find the range of values of $\arg z$ for points on this locus.** **3*i***



A and B are the points of least and greatest argument respectively.

Clearly B has argument $\frac{\pi}{2}$.

Find argument of A given that it is the point of contact of the tangent $y = mx$ and the parabola $(y-1)^2 = 4x$.

$$(y-1)^2 = 4 \cdot \frac{y}{m}$$

$$my^2 - 2my + m = 4y$$

$$my^2 - 2(m+2)y + m = 0$$

For tangent,

$$\begin{aligned} \Delta &= 0 \\ 4(m+2)^2 - 4m^2 &= 0 \\ \cancel{4m^2} + 16m + 16 - \cancel{4m^2} &= 0 \\ m &= -1 \end{aligned}$$

$$\therefore \quad -\frac{\pi}{4} \le \arg z \le \frac{\pi}{2}$$

Marker Comments

Drawing the correct parabola was essential to getting the range of arg$z$. This is 2U work.

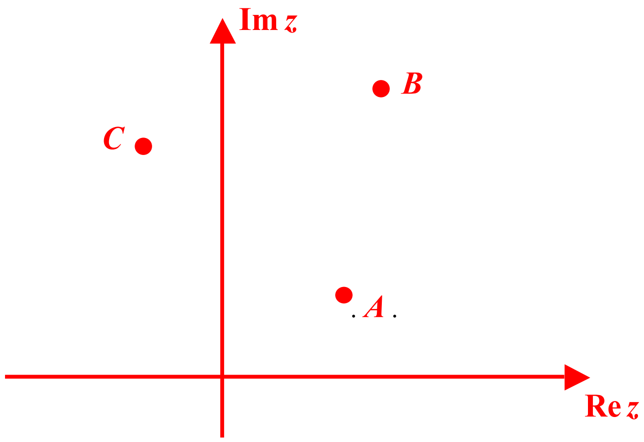
Many students realised that the minimum of arg$z$ occurred when a tangent to the parabola passes though the origin, but were unable to go any further.

Many students got the correct minimum but were unable to justify why.
This included those who found the tangent at the extreme end of the latus rectum. This was penalised.

Question 7 (continued)

(c)

The points A, B, and C, representing non-zero complex numbers a, b and c, are the vertices in anti-clockwise order of an equilateral triangle in the Argand diagram. **2*i***



By considering vectors AB and AC, show that the complex numbers a, b and c satisfy

$$2c = (a+b) + i\sqrt{3}(b-a)$$

$\overrightarrow{AC}$ corresponds to the complex number $c - a$.
$\overrightarrow{AB}$ corresponds to the complex number $b - a$.

As $\Delta\, ABC$ is an equilateral triangle, then $\overrightarrow{AC}$ is a rotation of $\overrightarrow{AB}$, anticlockwise, by 60°

$$\therefore c - a = (b-a)\operatorname{cis}\frac{\pi}{3}$$

$$c - a = (b-a)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$

$$2(c-a) = (b-a)(1+i\sqrt{3})$$

$$2c - 2a = (b-a) + i\sqrt{3}(b-a)$$

$$\therefore 2c = (a+b) + i\sqrt{3}(b-a)$$

Marker Comments

Some students are not following instructions i.e. "by considering vectors AB and AC …"
This just made it harder for them to be successful. They were not penalised for doing so.

Many students had $c - a = (b-a)\operatorname{cis}\frac{2\pi}{3}$ or writing $c - a = (b-a)\times i\frac{\sqrt{3}}{2}$ to rotate by $\frac{\pi}{3}$.

NB To rotate a vector by an angle of θ anticlockwise, then the corresponding complex number z is multiplied by $\operatorname{cis}\theta$.

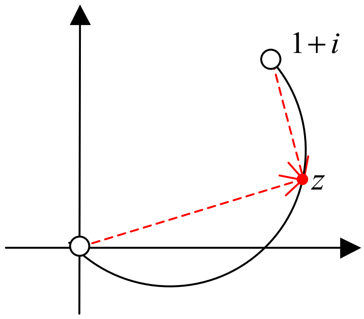
Question 7 (continued)

(d)

Certain complex numbers z satisfy the relation $cz = i(z-\omega)$ where $\omega = 1+i$ and $c > 0$ is a real number which is free to vary. **3**

On an Argand diagram, sketch the locus of all such complex numbers z.

A sketch only, without justification or development of the locus, will not earn full credit.



Method 1: Geometrically

$cz = i(z-\omega) \Rightarrow \arg cz = \arg\left[i(z-\omega)\right]$

$\therefore \arg c + \arg z = \arg i + \arg(z-\omega)$

$\therefore \arg z - \arg(z-\omega) = \dfrac{\pi}{2}$ $\qquad\left[\arg\left(\dfrac{z}{z-\omega}\right) = \dfrac{\pi}{2}\right]$

Since the angle in a semi-circle is $90°$, the locus of z is the semi-circle shown above.

Method 2a: Algebraically

Let $z = x + iy$

$cz = i(z-\omega) \Rightarrow cz = iz - i\omega$

$\therefore c(x+iy) = i(x+iy-1-i)$

$\therefore cx + icy = 1 - y + i(x-1)$

$\therefore cx + y - 1 + i(cy - x + 1) = 0$

Equating real and imaginary parts: $cx + y - 1 = 0$ and $cy - x + 1 = 0$

Now $cx + y - 1 = 0 \Rightarrow c = \dfrac{1-y}{x}$ and $cy - x + 1 = 0 \Rightarrow c = \dfrac{x-1}{y}$

$\therefore \dfrac{1-y}{x} = \dfrac{x-1}{y}$

$\therefore y(1-y) = x(x-1)$

$\therefore x^2 - x + y^2 - y = 0$

$\therefore \left(x-\tfrac{1}{2}\right)^2 + \left(y-\tfrac{1}{2}\right)^2 = \tfrac{1}{2}$

NB $c = \dfrac{1-y}{x}$, for $c > 0$ then $1 - y > 0$ and $x > 0$ i.e. $y < 1$ and $x > 0$

OR $1 - y < 0$ and $x < 0$ i.e. $y > 1$ and $x < 0$ BUT there is no place on the circle where this can occur as the y-intercepts are 0 and 1 and for $x < 0$, $y < 1$.

$\therefore$ $y < 1$ and $x > 0$ which is the semi-circle drawn above.

Method 2b: Algebraically

$$cz = i(z-\omega) \Rightarrow cz = iz - i\omega$$

$$\therefore z(c-i) = -i\omega = -i(1+i) = 1-i$$

$$\therefore z = \frac{1-i}{c-i}$$

$$z = \frac{1-i}{c-i} \times \frac{c+i}{c+i} = \frac{c+1+i(1-c)}{c^2+1}$$

Let $z = x + iy$ and so $x = \dfrac{c+1}{c^2+1},\ y = \dfrac{1-c}{c^2+1}$

$$\therefore x^2 + y^2 = \left(\frac{c+1}{c^2+1}\right)^2 + \left(\frac{1-c}{c^2+1}\right)^2$$

$$= \frac{2}{c^2+1}$$

$$= x + y$$

$$\therefore x^2 - x + y^2 - y = 0$$

$$\therefore \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$$

NB for $c > 0$

$$x = \frac{c+1}{c^2+1} \Rightarrow x > 0$$

$$y = \frac{1-c}{c^2+1} \Rightarrow y < 1$$

With $x > 0$ and $y < 1$ then the locus of z is the semi-circle shown above.

Marker Comments

A few students solved this using the methods shown above.

Students who used the algebraic methods were not able to get the necessary restrictions to the circle and were penalised.

End of solutions